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Statistical Analysis of Morphological Filters for a Stationary Source

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정상 신호원에 대한 형태론적 여파기의 통계 분석

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ABSTRACT

Statistical properties of three dual morphological filters of dilation/erosion, closing/opening, and clos-opening/open-closing and their statistical relations are analyzed for a stationary source using probability laws together with set operations. The principal statistical properties obtained are : (1) a mutual relation between the cumulative distribution functions (cdf' s) of filtered sequences by a pair of dual morphological filters, and (2) the representation of cdf s of filtered sequences by hybrid morphological filters as a linear sum of those of the basic dual filters, namely, dilation and erosion. From these, we can subsequently obtain some relative statistics of morphologically filtered sequences compared with the source statistics.

Computer simulations for a 1st-order Gauss-Markov source clearly show the validity of the statistical properties developed in the paper

要 約

정상 신호원에 대한 3가지 쌍대적 형태론적 여파기(불림/녹임, 붙임/열림 및 붙임-열림/열림-붙임)의 통계적 특성과 상호간의 통계적 관계를 확률법칙과 집합연산에 의해 분석하였다. 본 논문에서 얻은 주요특성은 둘로 요약할 수 있는데, 첫째, 쌍대되는 형태론적 여파기에 의해 여파된 신호의 누적분포함수간의 상호관계를 나타냈다는 것과, 둘째, 조합형 여파기에 의해 여파된 신호의 누적분포함수를 기본적인 불림 및 녹임 여파기에 의한 누적분포함수의 가중 합으로 나타냈다 것이다. 아울러 이러한 특성에서 신호원과 형태론적으로 여파된 신호와의 상대적인 통계적 관계를 얻을 수 있었다.

1차 Gauss-Markov 신호원에 대한 컴퓨터 모의실험결과가 본 논문에서 정립한 통계적 특성과 부합됨을 알 수 있었다.

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Ⅰ. Introduction

Mathematical morphology introduced by Matheron [1] and Serra [2] is one of the most active research fields in nonlinear signal processing, which is originated from the set algebra for the quantitative description of geometric structure of binary images and later on extended to grayscale morphology [2,3,4]. Morphological filters such as opening, open-closing, etc., have found extensive applications in the areas of image analysis and processing [5,6,7,8].

The basic dual morphological operations, i.e., the dilation and erosion, are completely equivalent to the 1st and last ranked-order filters, respectively. Their statistics for an independent and identically distributed(iid) source have been already found in some bibliographies about "median filter" [9] and about "order statistics" [10,11].

The other two duals, the opening/closing and open-closing/clos-opening, compared with the median filtering, possess very interesting properties [12,13] that the median filtered sequence is bounded below by opening and above by closing, the median root sequence is bounded below by open-closing and above by clos-opening, and the open-closed and clos-opened sequences are median roots themselves. For an iid source, statistical properties of the morphological open-closing and clos-opening have been analyzed in [6] with the threshold decomposition method, and statistics of all the three kinds of duals mentioned above are studied in [14]. Recently a new approach to statistical analysis of opening based on the basis functions of the filters has been considered in [15].

So far statistical analyses of the morphological filters have been widely studied for an iid source in [6,9,14,15] as noted above. However they have limited practical importance because real signal sources such as speech and image signals, unlike white noise, are far from the assumption of iid processes. From this point of view, we are interested in a stationary source.

The goal of this paper is to find statistical relations between dual morphological filters and to analyze their statistical properties for a stationary source.

The paper is organized into five sections. In section II, the dual morphological operations and their statistical relations for a stationary source are dealt. In section III, the statistical properties of dual morphological operations for a stationary source are developed. In section IV, for a specific application, the statistics are analyzed for a stationary source whose joint probability density function (pdf) is symmetrical about its mean. In section V, to test the statistical properties developed in preceding sections, computer simulations are carried out for a stationary 1st-order Gauss-Markov source.

Ⅱ. Dual morphological operations and their statistical relations

The basic dual morphological operations are dilation and erosion. The others are hybrid operations using the basic duals. Let us define these duals briefly and also examine their statistical relations.

1. Dual morphological operations

For a given random sequence $X=\{x_i : i\in I\}$ and a given structuring element $W=\{i : i\in I\}$, one dimensional grayscale morphological dilation and erosion of X by W, denoted $D_W\{X\}$ and $E_W\{X\}$, respectively, are defined as [7]

$D_W(X) = \{d_i : d_i = \max(x_{i+j} : j \in W)\}$ and
$E_W(X) = \{e_i : e_i = \min(x_{i+j} : j \in W)\}$,
where I is a set of integers.

Let X_j^T and $\underline{W}$, respectively, denote the sequence X translated by $j \in I$ and the reflected W with respect to the origin as

$X_j^T = \{x_{i+j} : i \in I,\ j \in I\}$ and
$\underline{W} = \{-i : i \in I\}$.

It then follows from the above definitions that

$$D_{W_j^T}\{X\} = D_W\{X_j^T\} = (D_W\{X\})_j^T,$$
$$E_{W_j^T}\{X\} = E_W\{X^T_{-j}\} = (E_W\{X\})^T_{-j}, \quad (1)$$
$$D_W\{X\} = -E_{\underline{W}}\{-X\}, \text{ and } E_W\{X\} = -D_{\underline{W}}\{-X\}.$$

The hybrid duals, the opening and closing of X by W, denoted $O_W\{X\} = \{o_i : i \in I\}$ and $C_W\{X\} = \{c_i : i \in I\}$, respectively, are defined by

$O_W\{X\} = D_W\{E_W\{X\}\}$ and $C_W\{X\} = E_W\{D_W\{X\}\}$.

The other hybrid duals, the open-closing(opening followed by a closing) and clos-opening, denoted $OC_W\{X\}$ and $CO_W\{X\}$, respectively, are defined by

$OC_W\{X\} = C_W\{O_W\{X\}\}$ and $CO_W\{X\} = O_W\{C_W\{X\}\}$.

It follows from the translation properties in Eq.(1) and structuring element decomposition that

$O_W\{X\} = -C_{\underline{W}}\{-X\}$, $O_{W_j^T}\{X\} = O_W\{X\}$,
$OC_W\{X\} = -CO_{\underline{W}}\{-X\}$, $C_{W_j^T}\{X\} = C_W\{X\}$,
$OC_W\{X\} = E_W\{D_V\{E_W\{X\}\}\}$, $OC_{W_j^T}\{X\} = OC_W\{X\}$,
$CO_W\{X\} = D_W\{E_V\{D_W\{X\}\}\}$, and
$CO_{W_j^T}\{X\} = CO_W\{X\}$,

where V is identical to the binary dilation result of W by itself, i.e.,

$V = \{i+j : i \in W,\ j \in W\}$.

So far we have defined the duals and described their functional relations. From the above three kinds of duals, we can systematically describe the duals as in the following property 1.

Property 1 : Let A and B be dual morphological operations such as dilation and erosion, opening and closing, and clos-opening and open-closing. Then the filtered X by dual morphological operations with a structuring element W can be expressed by

$A_W\{X\} = -B_{\underline{W}}\{-X\}$,

and if W is a contiguous structuring element of size N such that

$$W = \{i_o + i : i = 0, 1, \cdots, N-1 \text{ and } i_o \in I\} \quad (2)$$

then

$$A_W\{X\} = -(B_W\{-X\})_j^T, \quad (3)$$

where the translation j equals zero except for dilation and erosion, i.e.,

$D_W\{X\} = -(E_W\{-X\})^T_{N+2i_o-1}$.

Proof : The 1st statement is merely a summarized form of the above results and Eq.(3) is clear from Eq.(1) since $\underline{W} = W^T_{1-2i_o-N}$ for Eq.(2). □

In the remainder of the paper, W will denote a contiguous structuring element of size N as in Eq.(2) and W_{+k} will denote a contiguous structuring element of size $N+k$.

2. Cdf's of dual morphological operations for a stationary source

Theorem 1 : Let A and B be dual and X be a continuous random sequence whose cdf is continuous everywhere and stationary in the strict sense. Then the cdf s of $A_W\{X\}$and $B_W\{X\}$, denoted $F_A(x)$ and $F_B(x)$, respectively, have the following properties :

$1 - F_A(-x) = FB_{W\{-X\}}(x)$

where the right-hand side is the cdf of $B_W\{-X\}$ and therefore can be obtained by a probabilistic transformation of $F_B(x)$ corresponding to the change of source polarity.

Proof : This statement is obvious from the property 1, the stationarity of morphologically filtered sequences by the afore-said three duals for a stationary source [16], and the fact that the cdf s of Z and $-Z$ are $F_z(z)$ and $1-F_z(-z)$, respectively. □

For an iid source, the statistical properties of the three kinds of dual morphological operations have been already investigated [6,9,14,15], but here, for the verification of the duality property in theorem 1 and to point out some mistakes in [6],[13], and [14], the correct cdf s are summarized as follows.

$$F_D(x) = F_X(x)^N,$$
$$F_E(x) = 1-\bar{F}_X(x)^N,$$
$$F_C(x) = NF_X(x)^N-(N-1)F_X(x)^{N+1},$$
$$F_O(x) = 1-N\bar{F}_X(x)^N+(N-1)\bar{F}_X(x)^{N+1}, \quad (4)$$
$$F_{CO}(x) = F_C(x) + F_X(x)^{2N}\,\bar{F}_X(x)\,\Big(1 + \frac{(N-2)(N+1)}{2}\,\bar{F}_X(x)\Big)u(N-2),$$
$$F_{OC}(x) = F_O(x) - \bar{F}_X(x)^{2N}F_X(x)\,\Big(1 + \frac{(N-2)(N+1)}{2}\,F_X(x)\Big)u(N-2),$$

where $\bar{F}_X(x) = 1-F_X(x)$ and $u(N)$ is the unit step sequence.

For a simple verification of correctness of Eq.(4), we can substitute N=1 into Eq.(4) and then verify whether each distribution function equals $F_X(x)$ or not. If so, it is right because X is not affected by any morphological operation if N=1.It is then easy to show that the references mentioned above are incorrect for $F_{CO}(x)$ and $F_{OC}(x)$ for iid sources.

Now we define some probability laws useful for the remainder of the paper.

P-1 : $P(A)=1-P(\bar{A})$,

P-2 : $P(A+B)=P(A)+P(B)-P(A,B)$,

P-3 : $P(A+(A,B)+(A,B,C)+\cdots)=P(A)$,

$$P-4:\ P(A_1,A_2,\cdots,A_N)=\sum_{i=1}^{N}P(A_i) - \sum_{i=1}^{N-1}P(A_i+(A_{i+1},A_{i+2},\cdots,A_N))\,u(N-2),$$

$$P-5:\ P(A_1+A_2+\cdots+A_N)=\sum_{i=1}^{N}P(A_i) - \sum_{i=1}^{N-1}P(A_i,(A_{i+1}+A_{i+2}+\cdots+A_N))\,u(N-2),$$

P-6 : $P(A+B)\geq P(A)$, $P(A,B)\leq P(A)$,

P-7 : $P(A)-P(A,B)+P(A,B,C)-P(B,C) = P(A+B+C)-P(B+C)\geq 0$

P-8: $J_K(B_{i/M})\equiv(B_{i/M},(B_{i+1/M}+B_{i+2/M}+\ +B_{i+K/M})) = B_{i/M+1}$ for $1\leq K\leq M$

where $\bar{A}$ denotes the complement of an event A, $(A+B)$ and (A,B) the union and intersection of two events A and B, respectively. P-4 and P-5 result from the recursive operations of P-2 and satisfy the duality principle [17, pp.24], and in P-8 $B_{i/M}=(B_i,B_{i+1},\cdots,B_{i+M-1})$.

Ⅲ. Statistics of dual morphological operations for a stationary source

In the remainder of the paper we assume the source X to be stationary in the strict sense, and $A_{W+k}(X)$ denotes the filtered X by a morphological operation A using W_{+k} and $F_{A+k}(x)$ the cdf of $A_{W+k}(X)$.

Theorem 2 (dilation and erosion) : Let X be a stationary continuous random sequence. Then the dilated and eroded X's with W have the following properties :

(a) $F_{D+1}(x)\leq F_D(x)\leq F_X(x)\leq F_E(x)\leq F_{E+1}(x)$,

(b) $2F_D(x)\leq F_{D-1}(x)+F_{D+1}(x)$,
$2F_E(x)\geq F_{E-1}(x)+F_{E+1}(x)$.

Since dilation is max operation within structuring element, $D_W(X)\leq D_{W+1}(X)$. Thus it is reasonable that $F_{D+1}(x)\leq F_D(x)$ as in (a) of Theorem 2. Furthermore as N increases, the increasing rate of dilated result corresponding to the unit increment of size of structuring element from N to N+1 becomes smaller. Therefore $F_D(x)-F_{D+1}(x) \le F_{D-1}(x)-F_D(x)$ as in (b) of Theorem 2. In other words, the difference $F_D(x)-F_{D+1}(x)$ is decreasing function with respect to the ratio N/(N+1).

The results of erosion can be explained in a similar manner.

Theorem 3 (closing and opening) : Let X be a stationary continuous random sequence. Then the closed and opened X s with W have the following properties :

(a) $F_C(x)=NF_D(x)-(N-1)F_{D+1}(x)$,
$F_O(x)=NF_E(x)-(N-1)F_{E+1}(x)$.

(b) $F_{C+1}(x) \le F_C(x) \le F_X(x) \le F_O(x) \le F_{O+1}(x)$,
$F_D(x) \le F_C(x) \le F_X(x) \le F_O(x) \le F_E(x)$.

Note that the theorem 3 also holds for Eq.(4) of iid sources.

Theorem 4 (clos-opening and open-closing) : Let X be a stationary continuous random sequence. Then the clos-opened and open-closed X s with W have the following properties :

(a)
$$F_{CO}(x)=F_C(x)+[F_{D(N,1,N)}(x)-F_{D(N,0,N+1)}(x)]u(N-2)$$
$$+\sum_{i=2}^{N-2}(i+1)[F_{D(N,i+1,N)}(x)-F_{D(N,i,N+1)}(x)$$
$$+F_{D(N+1,i-1,N+1)}(x)-F_{D(N+1,i,N)}(x)]u(N-3),$$
$$F_{OC}(x)=F_O(x)+[F_{E(N,1,N)}(x)-F_{E(N+1,0,N)}(x)]u(N-2)$$
$$+\sum_{i=2}^{N-2}(i+1)[F_{E(N,i+1,N)}(x)-F_{E(N+1,i,N)}(x)$$
$$+F_{E(N+1,i-1,N+1)}(x)-F_{E(N,i,N+1)}(x)]u(N-3).$$

(b) $F_C(x) \le F_{CO}(x)$, $\quad F_{OC}(x) \le F_O(x)$.

where $F_{D(L,M,N)}(x)$ and $F_{E(L,M,N)}(x)$ are the cdf's of dilated and eroded X's using a non-contiguous structuring element $W_{(L,M,N)}$ which consists of two contiguous structuring elements of size L and N and split by M, defined by

$$W_{(L,M,N)}=\{1,2,\ ,L,L+M+1,L+M+2,\ ,L+M+N\}.$$

The proofs of above Theorems 2, 3, and 4 will be later in the appendix. Note that the cdf s of hybrid duals can be represented by a linear combination of those of the basic duals, namely, dilation and erosion.

Ⅳ. Statistics for stationary sources of symmetrical joint pdf

For a specific application of the results of the previous section III and for simple analysis, we assume that (1) the source X is a stationary continuous random sequence and then has a common pdf $f_X(x)$, i.e., the same marginal pdfs, with mean m_X and variance σ^2_X, (2) whose joint pdf is symmetric about the mean, i.e.,

$$f(m_X+x_1, m_X+x_2, \cdot\cdot\cdot, m_X+x_N)$$
$$= f(m_X-x_1, m_X-x_2, \cdot\cdot\cdot, m_X-x_N).$$

Note that a stationary jointly Gaussian random sequence does satisfy the above assumptions.

Theorem 5 : Let A and B be dual morphological operations and X a stationary continuous random sequence whose joint pdf is symmetric about the mean. Then the filtered X by dual operations using W have the following property of symmetry :

$$F_A(m_X+x)=1-F_B(m_X-x).$$

Proof : Consider first the basic duals of dilation and erosion. From their definitions, Theorem 1, and the definition of the cdf, $F_D(x)$ and $F_E(x)$ can be rewritten by

$$F_D(x)=\int_{-\infty}^{x}\int_{-\infty}^{x}\cdots\int_{-\infty}^{x}f(x_1,x_2,\cdots,x_N)d\bar{x},$$

$$F_E(x)=1-\int_{x}^{\infty}\int_{x}^{\infty}\cdots\int_{x}^{\infty}f(x_1,x_2,\cdots,x_N)d\bar{x}.$$

where $d\bar{x}=dx_1dx_2\cdots dx_N$. Thus we have

$$F_D(m_X+x)=\int_{-\infty}^{m_X+x}\int_{-\infty}^{m_X+x}\cdots\int_{-\infty}^{m_X+x}f(x_1,x_2,\cdots,x_N)d\bar{x}=\int_{-\infty}^{x}\int_{-\infty}^{x}\cdots$$

$$\int_{-\infty}^{x}f(m_X+x_1,m_X+x_2,\cdots,m_X+x_N)d\bar{x}$$

$$=\int_{-\infty}^{x}\int_{-\infty}^{x}\cdots\int_{-\infty}^{x}f(m_X-x_1,m_X-x_2,\cdots,m_X-x_N)d\bar{x}=\int_{m_X-x}^{\infty}\int_{m_X-x}^{\infty}\cdots$$

$$\int_{m_X-x}^{\infty}f(x_1,x_2,\cdots,x_N)d\bar{x}$$

$$=1-F_E(m_X-x),$$

where the third equality follows from the symmetry of the joint pdf.

Applying the above methodology and results of basic duals into Theorems 3 and 4, and using the fact that $F_{D(L,M,N)}(m_X+x)=1-F_{E(N,M,L)}(m_X-x)$, it is easy to see that $F_C(m_X+x)=1-F_O(m_X-x)$ and $F_{CO}(m_X+x)=1-F_{OC}(m_X-x)$. And so Theorem 5 is proved. □

Theorem 6 : Let X be a stationary 1st-order Gauss-Markov process, with zero mean, unit variance, and correlation coefficient ρ, generated by [17,18]

$$x_i=\rho x_{i-1}+w_i, \tag{5}$$

where wi is white Gaussian noise with zero mean and variance $(1-\rho^2)$ and independent of past outputs, i.e., $E(w_i\, x_{i-j})=0$ for $j>0$. Then for highly correlated process $(\rho^2\to1)$, the morphologically filtered X with $W_N(N\ge2)$ have the following statistical relationships.

(a) as $\rho\to1$,

$$F_D(x)=F_E(x)=F_C(x)=F_O(x)=F_{CO}(x)=F_{OC}(x)=F_X(x),$$

(b) as $\rho\to-1$,

$$F_D(x)=F_C(x)=F_{CO}(x)=(2F_X(x)-1)u(x),$$

$$F_E(x)=F_O(x)=F_{OC}(x)=(2F_X(x)-1)u(-x)+1,$$

(c) standardized central moments

ρ	mean	variance	skewness	kurtosis
$\lim_{\rho\to1}$	0	1	0	3
$\lim_{\rho\to-1}$	$\sqrt{\frac{2}{\pi}}$ ≒0.798	$1-\frac{2}{\pi}$ ≒0.363	$\frac{\sqrt{2}(4-\pi)}{(\pi-2)^{1.5}}$ ≒0.995	$\frac{3\pi^2-4\pi-12}{(\pi-2)^2}$ ≒3.87

Proof : If $\rho\to1$, then $x_i\to x_{i-1}$ and therefore all the cdf's $F_X(x)$ regardless of the types of morphological operations. And if $\rho\to-1$, then $x_i\to -x_{i-1}$, i.e., x_i is an alternative sequence with respect to the index i. Thus we have

$$\lim_{\rho\to-1}F_D(x)=P(\max(x_0,-x_0)\le x)$$
$$=P(0\le x_0\le x)+P(-x\le x_0<0)$$
$$=(2F_X(x)-1)u(x),$$
$$\lim_{\rho\to-1}F_E(x)=P(\min(x_0,-x_0)\ x)$$
$$=1-P(\min(x_0,-x_0)>x)$$
$$=1-P(0\le x_0<-x)-P(x<x_0<0)$$
$$=1+(2F_X(x)-1)u(-x).$$

Using the fact that $\lim_{\rho\to-1}F_{D(N,i,N)}(x)=\lim_{\rho\to-1}F_{D(N,i\pm1,N\mp1)}(x)$ and substituting the above relations into theorems 3 and 4, we find (a) and (b) of Theorem 6. And we obtain (c) of Theorem 6 from the definitions that, for a random variable Z, the mean $m=E(Z)$, the variance $\sigma^2=E((Z-m)^2)$, the skewness=E $((Z-m)^3)/\sigma^3$, and the kurtosis=E $((Z-m)^4)/\sigma^4$. Among these the skewness and kurtosis are sometimes used as dimensionless measures of asymmetry and sharpness of pdf, respectively. □

V. Simulations

To test the statistical properties of morphological operations developed in previous sections II, III, and IV, we have generated eleven sources of 1st-order Gauss-Markov process varying correlation coefficient in Eq.(5) as ρ=0, ±0.3, ±0.5, ±0.7, ±0.9, ±0.99. Each of them consists of 2 x 10^6 samples. For these sources, we have computed numerically the statistics of three kinds of dual morphological operations varying the size of contiguous structuring element as N = 2, 3, · · ·, 10.

Fig. 1 shows simulated cdf's and pdf's of morphologically filtered sequences by three extensive operations of dilation, closing, and clos-opening using W_5, when ρ=0.3, where, for simplicity, those of their dual antiextensive operations are omitted since they can be got from the properties of symmetry of theorem 5. We can thus find in Fig. 1 that as in Theorems 2, 3, and 4, $F_C(x) \le F_{CO}(x)$, $F_{OC}(x) \le F_O(x)$, and $F_D(x) \le F_C(x) \le F_X(x) \le F_O(x) \le F_E(x)$. Note that $m_{CO} \ge m_X \ge m_{OC}$ even if $F_{CO}(x) \nleq F_X(x) \nleq F_{OC}(x)$.

Table 1 shows the mean values of the filtered sequences by the extensive morphological operations versus N when ρ=0.3. Here we can see that the simulated means hold for Theorems 2, 3 and 4. For example, we have

Table 1. Simulated mean values versus N for ρ = 0.3.

N	Dilation	Closing	Clos-opening
2	0.4717	0.2038	0.0972
3	0.7397	0.3726	0.2610
4	0.9232	0.5091	0.4028
5	1.0612	0.6227	0.5225
6	1.1708	0.7203	0.6253
7	1.2609	0.8017	0.7102
8	1.3374	0.8751	0.7880
9	1.4035	0.9410	0.8583
10	1.4613	0.9987	0.9193

the following numerical results for N = 5 as

$2m_D - m_{D+1} - m_{D-1} = 2 \times 1.0612 - 1.1708 - 0.9232 > 0$,

$m_C - 5m_D + 4m_{D+1} = 0.6227 - 5 \times 1.0612 + 4 \times 1.1708 \doteqdot 0$,

$m_C - m_{CO} = 0.6227 - 0.5225 > 0$.

Fig. 2 represents the simulated statistics of the filtered sequences by the extensive morphological operations versus N for ρ=0.9 and ρ = -0.99. Their means and variances are monotonic with respect to N, while the skewness and kurtosis are not monotonic. As expected from Theorem 6, for a positively higher ρ, the morphologically filtered sequences have almost gaussian-like statistics in view of skewness≑0 and kurtosis≑3, while for a negatively higher they become half truncated gaussian-like with both skewness≑0.995 and kurtosis≑3.87.

Similarly, Fig. 3 illustrates the results versus for N=5. All the statistics converge on the two extreme limits of $\rho \to \pm 1$, as tabulated in (c) of Theorem 6.

Ⅵ. Conclusions

In this paper, We have analyzed statistical

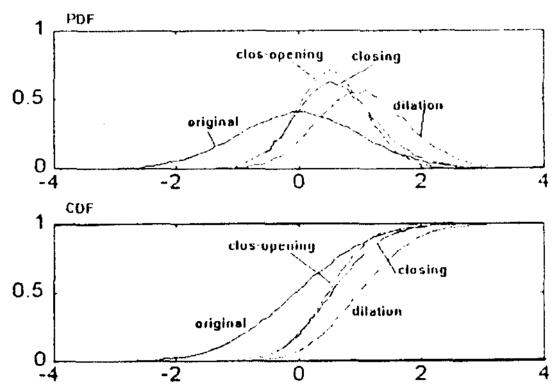

Fig. 1. Simulated cdr's and pdf's of morpologically filtered sequences by three extensive operations of dilation, closing and clos-opening with W_5 for a 1st-order Gauss-Markow source of ρ=0.3

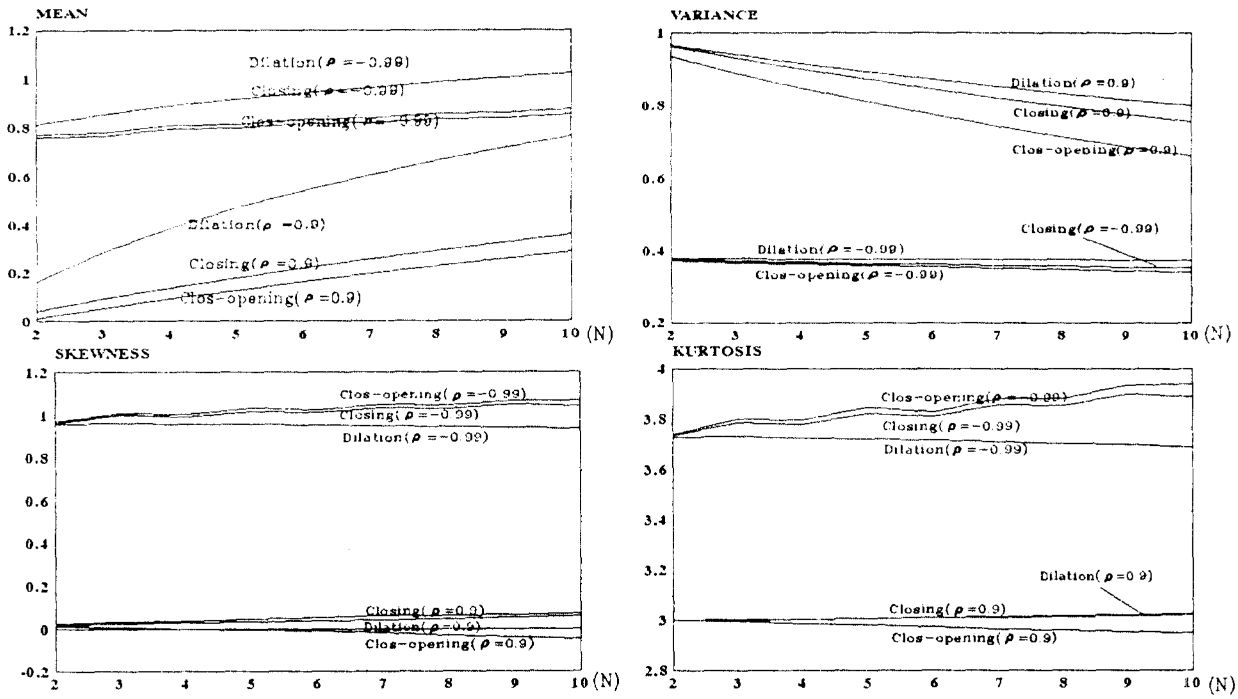

Fig. 2. Simulated mean values of the filtered sequences versus N for a lst-order Gauss-Markov source of ρ=0.9 and ρ=-0.99.

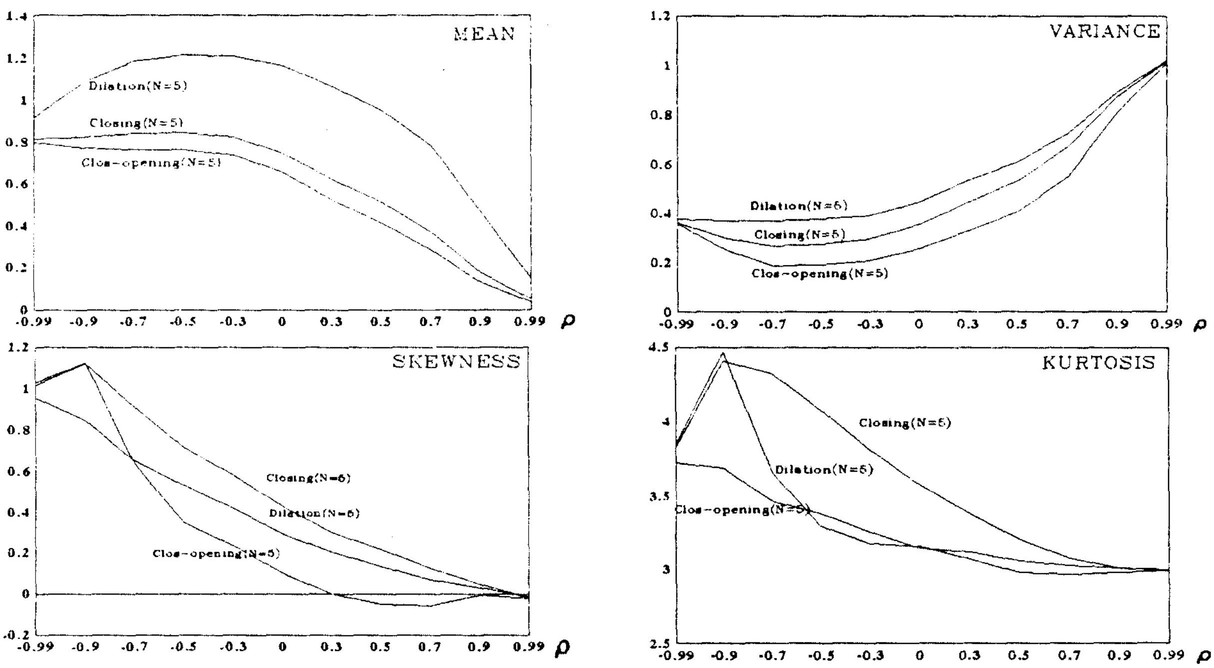

Fig. 3. Simulated statistics of morphological filtered sequences with W_5 versus the correlation coefficient(ρ) of a lst-order Gauss-Markov source.

relations and properties of dual morphological filters for a stationary source using some probability laws and set operations.

We first introduced statistical relations of the basic dual of dilation and erosion as stated in theorem 2. Based on these, we found out the cdf s of the hybrid morphological operations, i.e., closing/opening and clos-opening/open-closing, in a closed form, as in theorems 3 and 4, and compared the cdf's and their means one another as in theorems 3 and 4.

For a specific source of symmetric joint pdf about its means, we saw, as expected, that the statistical symmetries also exist between duals as in theorem 5. As such a source and for the experimental purpose, we took a 1st-order Gauss-Markov source which are often referred to as a simple signal model in digital signal processing.

Computer simulations are carried out to see the statistical behaviors of the dual morphological filters for a 1st-order Gauss-Markov source, and the results show a good agreement with the theoretical statistical properties developed in this paper.

Appendix

In this section we present proofs of Theorems 2, 3, and 4 in section III.

1. Proof of Theorem 2 : Using the definition of dilation in section II, we have

$$F_D(x)=P(x_1\le x, x_2\le x, \cdots, x_N\le x)$$
$$\le P(x_1\le x, x_2\le x, \cdots, x_{N-1}\le x)\le P(x_1\le x),$$

where the 1sd equality follows from the stationarity [16], the last two inequalities from P-6, which equal $F_{D-1}(x)$ and $F_X(x)$, respectively. And using P-4 and P-8 and letting $X_i=(x_i\le x)$, $F_D(x)$ can be rewritten as

$$F_D(x)=F_X(x)+F_{D-1}(x)-P(X_1+X_{2/N-1}).$$

With recursive operations of this result and using P-6, we obtain the 1st term of (b) of Theorem 2.

The other properties of erosion can be proved in a similar manner. □

2. Proof of Theorem 3 : Using the definition of the opening in section II, letting $B_i=(x_i)x$ and $e_i=\min(x_{i+j}: j\in W)$, the cdf of $O_W(X)$ can be expressed as

$$F_O(x)=P(e_1\le x, e_2\le x, \cdots, e_N\le x)$$
$$=\sum_{i=1}^{N} P(e_i\le x)\sum_{i=1}^{N-1}(1-P(J_{N-i}(B_i/_N)))u(N-2)$$
$$=NF_E(x)-(N-1)F_{E+1}(x),$$

where the 1st equality follows from the stationarity [16] and the last two from P-4, P-8, and $P(B_{i/N+1})=1-F_{E+1}(x)$. And for closing, $F_C(x)$ can be obtained either by the same way used above or by using Theorem 1.

And from Theorem 2 and (a) of Theorem 3, it is clear that

$$F_{C+1}(x)-F_C(x)=N(2F_{D+1}(x)-F_{D+2}(x)-F_D(x))\le 0,$$
$$F_{O+1}(x)-F_O(x)=N(2F_{E+1}(x)-F_{E+2}(x)-F_E(x))\ge 0,$$
$$F_C(x)-F_D(x)=(N-1)(F_D(x)-F_{D+1}(x))\ge 0,$$
$$F_O(x)-F_E(x)=(N-1)(F_E(x)-F_{E+1}(x))\le 0.$$

And so Theorem 3 is proved. □

3. Proof of Theorem 4 : Using the definition of the clos-opening in section II, letting $D_i=(d_i)x$ where $d_i=\max(x_{i-j}: j\in W)$, and using $e_i=\min(d_{i+j} : j\in V,\ V=\{2i_o, 2i_o+1, \cdots, 2(i_o+N-1)\})$, we have

$$F_{CO}(x)=P(e_1\le x, e_2\le x, \cdots, e_N\le x)$$
$$=\sum_{i=1}^{N} P(e_i\le x)-\sum_{i=1}^{N-1}(1-P(J_{N-i}(D_{i/2N-1})))u(N-2) \quad (A.1)$$
$$=1-NP(D_{i/2N-1})+(N-1)P(D_{i/2N}),$$

where the 1st equality follows from stationarity [16] and the last two equalities from P-1, P-4, and P-8. Eq.(A.1) can be viewed as a weighted sum of $P(D_{i/K})$ with respect to K s equal to 2N and 2N-1.

Letting $X_i=(x_i \le x)$ and using P-4 and O-8, we have

$$P(D_{i/K})=1-KF_D(x)+R_K u(K-2), \qquad (A.2)$$

where $R_K=\sum_{i=1}^{K-1}=P(J_{K-i}(X_{i/N}))$

and can be rearranged from P-2 and P-8 :

$$R_K=\sum_{i=K-N}^{K-1} P(X_{i/N+1})u(K-N-1)$$

$$+\sum_{i=1}^{K-N-1} P(X_{i/N+1}+(X_{i/N}, s(1)))u(K-N-2) \qquad (A.3)$$

$$=(K-1)F_{D+1}(x)u(K-N-1)+Q_K,$$

where

$$s(n)=(X_{i+N+n/N}+X_{i+N+n+1/N}+\cdots+X_{K/N})u(K-N-n-i),$$

$$Q_K=\sum_{a=1}^{1}(-1)^a\sum_{i=1}^{K-N-1} P(X_{i/N+a}, s(1))u(K-N-2).$$

With some tedious and complex work, we have

$$Q_K=\sum_{a=0}^{1}\sum_{b=0}^{1}(-1)^{a+b}\sum_{K=1}^{K-N-1-b}(K-N-k-b)F_{D(N+b, k-a, N+a)}$$

$$(x)u(K-N-b-2).$$

Substituting Q_K into Eq.(A.3) and subsequently Eq.(A.3) into Eqs.(A.2) and (A.1), finally we obtain $F_{CO}(x)$ of (a) in Theorem 4, and also $F_{OC}(x)$ using Theorem 1.

The terms inside the 1st and 2nd brackets of $F_{CO}(x)$ in Theorem 4 equal $P(A)-P(A,B)$ and P(C+D+E)-P(D+E), respectively, with $A=(X_{1/N}, X_{N+2/N})$, $B=X_{N+1}$, $C=(X_{1/N}, X_{N+i+2/N})$, $D=X_{N+1}$, and $E=X_{N+i+1}$. Using P-6 and P-7, $F_{CO}(x)\ge F_C(x)$. Similarly it is readily verified that the terms inside the brackets of $F_{OC}(x)$ are all less than or equal to zero. And so Theorem 4 is proved. □

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