이변량 가우시안 Q-함수의 Craig 표현에 대한 기하학적인 유도

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A Geometric Derivation of the Craig Representation for the Two-Dimensional Gaussian Q-Function

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요 약

본 논문에서는 기하학적인 관점으로 이변량 가우시안 *Q*-함수의 Craig 표현에 대한 새롭고 간단한 유도를 제시하고 있다. 또한, 이러한 기하학적인 유도는 이변량 가우시안 *Q*-함수의 또 다른 Craig 표현 식을 제시하고 있다. 새롭게 유도된 이변량 가우시안 *Q*-함수의 Craig 식은 2개의 상관 가우시안 잡음에서 직교좌표의 변환으로 생성되는 2개 웨지 영역의 기하학으로부터 새롭게 구한 것이다. 제시된 Craig 형태는 이변량 가우시안 *Q*-함수로 표현되는 확률을 계산하는데, 중요한 역할을 할 수 있다.

Key Words: Geometric Derivation, Two-dimension, Craig, Gaussian *Q*-function, Cartesian

ABSTRACT

In this paper, we present a new and simple derivation of the Craig representation for the two-dimensional (2-D) Gaussian *Q*-function in the viewpoint of geometry. The geometric derivation also leads to an alternative Craig form for the 2-D Gaussian *Q*-function. The derived Craig form is newly obtained from the geometry of two wedge-shaped regions generated by the rotation of Cartesian coordinates over two correlated Gaussian noises. The presented Craig form can play a important role in computing the probability represented by the 2-D Gaussian *Q*-function.

Ⅰ. Introduction

It is very important to evaluate error probability performance in designing wireless communication systems. The Craig representation has played a key role when evaluating the error probability performance of digital modulation systems over fading channels by using the moment-generating function(MGF) approach[1-3]. Several derivations of the Craig form for the one-dimensional(1-D) Gaussian *Q*-function were recently reported[4-5]. The Craig form for the 2-D Gaussian *Q*-function applied to compute the error probability of *M*-ary phase shift keying system over various fading channels[6]~[7]. The approximation for the 2-D Gaussian *Q*-function was presented in terms of the 1-D Gaussian *Q*-function[8]. The algebraic derivation of the Craig form for the 2-D Gaussian *Q*-function was presented

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in terms of the change of variables[9]. However, it is said that there may exist a simple and new derivation of the Craig form.

It is well-known that the geometric derivation is easy to understand. Thus, in this paper, we will present a geometric derivation of the Craig representation by using the rotation of Cartesian coordinates. We hope that the presented derivation is easy and simple to understand the Craig form for the 2-D Gaussian *Q*-function.

Ⅱ. Problem

Our starting point is the analytical expression of a double integral for the 2-D Gaussian *Q*-function in the following:

$$Q(x,y;\rho)=\int_x^\infty\int_y^\infty\frac{\exp\left[-\frac{u^2-2\rho uv+v^2}{2(1-\rho^2)}\right]}{2\pi\sqrt{1-\rho^2}}dudv \qquad (1)$$

where ρ represents the correlation coefficient. Chronologically, Simon first derived the Craig form of the 2-D Gaussian *Q*-function by using the clever change of variables[9]:

$$\theta=\tan^{-1}\left(\frac{\tan\Phi\pm\rho}{\sqrt{1-\rho^2}}\right). \qquad (2)$$

It is also known that the Craig representation of the 2-D Gaussian *Q*-function developed by Simon is given by [9, eq. (10)]

$$\begin{aligned}Q(x,y;\rho)=&\frac{1}{2\pi}\int_0^{\tan^{-1}[(\sqrt{1-\rho^2}x/y)/(1-\rho x/y)]}\exp\left(-\frac{x^2}{2\sin^2\theta}\right)d\theta\\&+\frac{1}{2\pi}\int_0^{\tan^{-1}[(\sqrt{1-\rho^2}y/x)/(1-\rho y/x)]}\exp\left(-\frac{y^2}{2\sin^2\theta}\right)d\theta;\\&x\ge 0,y\ge 0.\end{aligned} \qquad (3)$$

where

$$\tan^{-1}\left(\frac{x}{y}\right)=\frac{\pi}{2}[1-sgn(y)]+sgn(y)\tan^{-1}\left(\frac{x}{|y|}\right)$$

in which $sgn(u)=1$ if $u\ge 0$ and $sgn(u)=-1$ if $u<0$. The generic Craig form for the 2-D Gaussian *Q*-function provided in [10] was derived from the upper limit of (3) and the properties of the 2-D Gaussian *Q*-function given in [11, eq. (26.3.8) and (26.3.9)]. So far, however, geometric interpretations on the change of variables (1) have not been reported in detail.

It is said that the geometric solution of a problem is easy and simple. Thus, in this letter, motivated by unknown geometric derivation of the Craig form for the 2-D Gaussian *Q*-function, we present a new derivation for the Craig form on the basis of the geometry of two wedge-shaped regions. The regions are generated by the rotation of Cartesian coordinates.

Ⅲ. Geometric Derivation of the Craig form for the 2-D Gaussian Q-Function

We consider X and Y to be two-dimensional Gaussian random variables (RV) with two zero means, $\mu_X=\mu_Y=0$, two unit variances, $\sigma_X^2 = \sigma_Y^2=1$ and a correlation coefficient, ρ_{XY}. Figure 1 shows a graphical representation of the open region, $\Omega=\{(x,y)|X\ge x^*,\ Y\ge y^*\}$, determined by two constants $x^*,y^*\ge 0$.

Here, as illustrated in Figure 1, we rotate the Cartesian coordinates counterclockwise through the angel $\psi=\tan^{-1}(y^*/x^*)$ about the origin in a way that

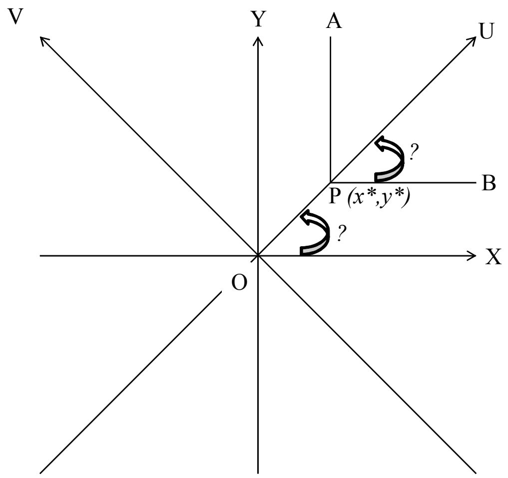

그림 1. 직교좌표 회전에 의해 생성된 2개의 웨지영역에 대한 기하적인 해석
Fig. 1. The geometric interpretation on two wedge-shaped regions generated by the rotation of Cartesian coordinates.

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}. \tag{4}$$

In terms of $U-V$ Cartesian coordinates, the open region, $\Omega = \{(x,y)|X \geq x^*,\ Y \geq y^*\}$, can be divided into two wedge-shaped regions, $\measuredangle APU$ and $\measuredangle BPU$. The probability of the wedge-shaped region, $\Pr\{\measuredangle APU\}$, is obtained by using (4) and the theory of linear combination of Gaussian RVs as

$$\begin{aligned} \Pr\{\measuredangle APU\}] &= \Pr\{X \geq x^*,\ V \geq 0\} \\ &= Q(x^*, 0; \rho_{XV}) \end{aligned} \tag{5}$$

where

$$\rho_{XV} = \frac{-\sin\psi + \rho_{XY}\cos\psi}{\sqrt{1-\rho_{XY}\sin 2\psi}}. \tag{6}$$

Similarly, the probability $\Pr\{\measuredangle BPU\}$ is obtained as

$$\Pr\{\measuredangle BPU\}] = \Pr\{Y \geq y^*,\ V < 0\} \tag{7}$$
$$= \int_{y^*}^{\infty}\int_{-\infty}^{0} \frac{\exp\left[-\frac{y^2 - 2\rho_{YV}yv + v^2}{2(1-\rho_{YV}^2)}\right]}{2\pi\sqrt{1-\rho_{YV}^2}} dv dy$$

where

$$\rho_{YV} = \frac{\cos\psi - \rho_{XY}\sin\psi}{\sqrt{1-\rho_{XY}\sin 2\psi}}. \tag{8}$$

Employing [11, eq.(26.3.6)] to (7) gives

$$\Pr\{\measuredangle BPU)\} = Q(y^*, 0; -\rho_{YV}). \tag{9}$$

Next, applying [12, eq. (A-5)] to (5) and (9), respectively, and using the trigonometric identity for $\sin^{-1}\phi + \cos^{-1}\phi = \pi/2$ yield the result in the Craig form as

$$\begin{aligned} Q(x^*, y^*; \rho) &= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}+\sin^{-1}\rho_{XV}} \exp\left(-\frac{x^{*2}}{2\sin^2\theta}\right) d\theta \\ &\quad + \frac{1}{2\pi}\int_0^{\cos^{-1}\rho_{YV}} \exp\left(-\frac{y^{*2}}{2\sin^2\theta}\right) d\theta; \\ &\quad x^* \geq 0,\ y^* \geq 0. \end{aligned} \tag{10}$$

Finally, letting $x^* = x$ and $y^* = y$ and substituting $\cos\psi = x/\sqrt{x^2+y^2}$ and $\sin\psi = y/\sqrt{x^2+y^2}$ into (10) result in an alternative expression for the Craig form presented in (3) as

$$\begin{aligned} Q(x, y; \rho) &= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}+\sin^{-1}\left(\frac{\rho x - y}{\sqrt{x^2 - 2\rho xy + y^2}}\right)} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta \\ &\quad + \frac{1}{2\pi}\int_0^{\cos^{-1}\left(\frac{x - \rho y}{\sqrt{x^2 - 2\rho xy + y^2}}\right)} \exp\left(-\frac{y^2}{2\sin^2\theta}\right) d\theta; \\ &\quad x \geq 0,\ y \geq 0. \end{aligned} \tag{11}$$

Note that the alternative expression of (11) do not require the user-defined arc tan function such as the upper limit of (3).

Ⅳ. Conclusion

The main contribution of this paper is the geometric derivation of the Craig form for the 2-D Gaussian *Q*-function by using the rotation of Cartesian coordinates. The new derivation leads to the alternative Craig representation of the 2-D Gaussian *Q*-function with geometric interpretation. The derived expression can be applicable to the exact computation of the probability represented by the 2-D Gaussian *Q*-function.

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