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Oblique Iterative Hard Thresholding 알고리즘을 이용한 압축 센싱의 보장된 Sparse 복원

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Guaranteed Sparse Recovery Using Oblique Iterative Hard Thresholding Algorithm in Compressive Sensing

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요 약

압축 센싱에서 측정 행렬 A의 $3s$-Restricted Isometry Constant가 $1/2$ 혹은 $1/\sqrt{3}$보다 작다면 모든 s-Sparse 벡터 $x \in R^N$는 측정 벡터 $y=Ax$ 또는 잡음이 섞인 벡터 $y=Ax+e$로부터 Iterative Hard Thresholding (IHT) 알고리즘에 의해 복원될 수 있다. 하지만, 이러한 복원은 신호 획득 기법의 특정한 가정 하에서 실질적인 알고리즘들에 의해 보장된다. 복원을 위한 핵심적인 가정 중에 하나는 측정 행렬이 Restricted Isometry Property (RIP)를 만족해야만 하는 것인데, 이 조건은 압축 센싱의 실제 응용 환경에서 종종 만족되지 않는다. 본 논문에서는 이방성 (Anisotropic) 경우에서 Restricted Biorthogonality Property (RBOP)로 불리는 RIP의 일반화와 Oblique Pursuit으로 불리는 새로운 복구 알고리즘들을 분석한다. 또한, IHT 알고리즘들을 위해 Restricted Biorthogonality Constant의 관점에서 성공적인 Sparse 신호 복원에 대한 분석을 제시한다.

Key Words : Compressive Sensing, Biorthogonality, Oblique Projection, Restricted Isometry Property, Iterative Hard Thresholding

ABSTRACT

It has been shown in compressive sensing that every s-sparse $x \in R^N$ can be recovered from the measurement vector $y=Ax$ or the noisy vector $y=Ax+e$ via ℓ_1-minimization as soon as the $3s$-restricted isometry constant of the sensing matrix A is smaller than $1/2$ or smaller than $1/\sqrt{3}$ by applying the Iterative Hard Thresholding (IHT) algorithm. However, recovery can be guaranteed by practical algorithms for some certain assumptions of acquisition schemes. One of the key assumption is that the sensing matrix must satisfy the Restricted Isometry Property (RIP), which is often violated in the setting of many practical applications. In this paper, we studied a generalization of RIP, called Restricted Biorthogonality Property (RBOP) for anisotropic cases, and the new recovery algorithms called oblique pursuits. Then, we provide an analysis on the success of sparse recovery in terms of restricted biorthogonality constant for the IHT algorithms.

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Ⅰ. Compressive Sensing

Compressive Sensing (CS) is a new approach to data acquisition, which allows to reconstruct a signal from fewer sample or a smaller number of measurements than Nyquist's rate, as long as the signal is sparse and the measurement is incoherent [1]. The process of acquiring compressed measurements is called the sensing and the recovering the original sparse signal from compressed measurements is called the reconstruction. Let $z_0 \in \boldsymbol{R}^N$ be an unknown signal, and $\boldsymbol{A} \in \boldsymbol{R}^{m \times N}(m < N)$ be the measurement matrix. Then, the measurement is expressed as

$$y = \boldsymbol{A}z_0 + w, \tag{1}$$

where $w \in \boldsymbol{R}^m$ denotes the additive noise.

In the conventional paradigm, CS aims to reconstruct signals that are approximately s-sparse over a dictionary $\boldsymbol{D} \in \boldsymbol{R}^{N \times N}$ from the compressed measurements $\boldsymbol{y}$. Let $\boldsymbol{x} \in \boldsymbol{R}^N$ be the coefficient vector of z_0 over $\boldsymbol{D}$ such that $z_0 \approx \boldsymbol{D}\boldsymbol{x}$ with $\boldsymbol{x}$ being s-sparse (i.e., $\|\boldsymbol{x}\|_0 \le s \ll N$). Then, the composition $\boldsymbol{\Psi} = \boldsymbol{A}\boldsymbol{D}$ can be viewed as a sensing matrix for $\boldsymbol{x}$ that produces the compressed measurement vector $\boldsymbol{y}$.

$$y = \boldsymbol{A}\boldsymbol{D}\boldsymbol{x} + w = \boldsymbol{\Psi}\boldsymbol{x} + w. \tag{2}$$

The Basis Pursuit (BP) problem in CS is to find the sparsest solution which has a minimal number of nonzero components by solving the following convex optimum problem.

$$\min_{\boldsymbol{x} \in \boldsymbol{R}^N} \|\boldsymbol{x}\|_1 \,\text{such that}\, \|\boldsymbol{\Psi}\boldsymbol{x} - \boldsymbol{y}\|_2 \le \epsilon. \tag{$P_{1,\epsilon}$}$$

There are basically two approaches to solve $(P_{1,\epsilon})$. First, we can exactly recover $\boldsymbol{x}$ via ℓ_1-minimization by solving the problem $(P_{1,\epsilon})$. The second method is using greedy algorithms for ℓ_1-minimization, such as Subspace Pursuit (SP), Compressive Sensing Matching Pursuit (CoSaMP), Orthogonal Matching Pursuit (OMP), Iterative Hard Thresholding (IHT), Hard Thresholding Pursuit (HTP) or their modifications[2-7].

The solution of $(P_{1,\epsilon})$ is guaranteed to be unique as soon as the measurement matrix Ψ satisfies the following s-Restricted Isometry Property (s-RIP).

$$(1-\delta_s)\|\boldsymbol{x}\|_2^2 \le \|\boldsymbol{\Psi}\boldsymbol{x}\|_2^2 \le (1+\delta_s)\|\boldsymbol{x}\|_2^2, \forall \|\boldsymbol{x}\|_0 \le s. \tag{3}$$

The smallest number δ_s is called s-Restricted Isometry Constant (s-RIC) of order s.

The rest of this paper is organized as follows. Section II introduces the CS problem and the motivation of this paper. In Section III, we present the oblique compressive sensing for anisotropic cases and some highlighted results in terms of reconstruction algorithms. Section IV provides a analysis on oblique iterative hard thresholding algorithm, which is a modification of iterative hard thresholding one in anisotropic cases. Section V concludes this paper.

Notations: We denote I_N as $N \times N$ identity matrix. $(\cdot)^*$ denotes the transpose operation, $\|\cdot\|_2$ denotes the ℓ_2-norm, $|T| = \text{supp}(T)$ denotes the number of elements in a given set T, and $E(\cdot)$ denotes the expectation operator. The ℓ_p-norm of a vector $\boldsymbol{x} = [x_1 \cdots x_N]$ is defined as $\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^{N} |x_i|^p\right)^{\frac{1}{p}}$. Note also that $\|\boldsymbol{x}\|_0 = \text{supp}(\boldsymbol{x}) = \#\{i : x_i \ne 0, i = 1, \cdots, N\}$. We call a signal $\boldsymbol{x}$ is a s-sparse vector if $\|\boldsymbol{x}\|_0 \le s$. We denote S as the index set satisfying $S \subseteq \{1, \cdots, N\}$ and $\overline{S} = \{1, \cdots, N\} \setminus S$ as its complement.

Ⅱ. Compressive Sensing Problem and Motivation

We consider a fixed vector $y \in \boldsymbol{R}^m$ as a measurement vector which is taken from m linear measurements of an unknown signal $\boldsymbol{x} \in \boldsymbol{R}^N$ for some $m \times N (m \ll N)$ sensing matrix $\boldsymbol{\Psi}$. It has been shown that if $\delta_{2s} < \sqrt{2} - 1$, the signal $\boldsymbol{x}$ can be perfectly recovered from noise-free measurements [8]. More conditions on the RIC values can be found in [9-11]. The RIP is a sufficient and necessary condition which guarantees unique and exact reconstruction of sparse signal via ℓ_1-minimization. The key assumptions of RIP guaranteed recovery are (i) isotropy property $E(\boldsymbol{\Psi}^*\boldsymbol{\Psi}) = \boldsymbol{I}_N$, and (ii) incoherence $\max_{i,k}|(\boldsymbol{\Psi})_{i,k}| \leq s$. However, in many practical applications, the isotropy property is not negligible as well as finding a small s-RIC. An example for those cases has been shown in [12]. Moreover, greedy recovery algorithms which are computationally efficient alternative solutions have been studied intensively in the isotropic cases, while their performance in the anisotropic case has not been much studied yet.

To tackle these issues, a generalized RIP called Restricted Biorthogonal Property (RBOP) and some modified versions of greedy algorithms called oblique greedy pursuits have been proposed [12]. The Oblique Iterative Hard Thresholding (ObIHT) is one of oblique pursuit algorithms with RBOP-based guarantees. It can be described as follows. From a reasonable guess for x, they start with a vector x_0 satisfying $\boldsymbol{\Psi}\boldsymbol{x}_0 = \boldsymbol{y}$. For a given iterate x^n, they generate a new iterate x^{n+1} such as

$$\boldsymbol{x}^{n+1} = H_s\left(\boldsymbol{x}^n + \widetilde{\boldsymbol{\Psi}}^*(\boldsymbol{y} - \boldsymbol{\Psi}\boldsymbol{x}^n)\right), \tag{4}$$

where $H_s(\cdot)$ is a nonlinear operation, which keeps s largest entries of a vector and set other ones to zeros and $\widetilde{\boldsymbol{\Psi}}$ is the matrix that satisfies RBOP and be discussed in the next section. They also update

$$T^n = \mathrm{supp}\left\{H_s\left(\widetilde{\boldsymbol{\Psi}}^*(\boldsymbol{y} - \boldsymbol{\Psi}\boldsymbol{x}^n)\right)\right\}. \tag{5}$$

$$U^n = T^n \cup S^n, \text{ where } S^n = \mathrm{supp}(\boldsymbol{x}^n). \tag{6}$$

$$\boldsymbol{u}^n = \arg\min\left\{\|\widetilde{\boldsymbol{\Psi}}^*(\boldsymbol{y} - \boldsymbol{\Psi}\boldsymbol{z})\|_2,\ \mathrm{supp}(\boldsymbol{z}) \subseteq U^n\right\}. \tag{7}$$

$$\boldsymbol{x}^{n+1} = H_s(\boldsymbol{u}^n). \tag{8}$$

The road map of finding solutions for the anisotropic cases can be summarized as follows. First, we extend the RIP of $\boldsymbol{\Psi}^*\boldsymbol{\Psi}$ to RBOP of $\widetilde{\boldsymbol{\Psi}}^*\boldsymbol{\Psi}$. Second, we construct the matrix $\widetilde{\boldsymbol{\Psi}}$ that satisfies RBOP, i.e., $E(\widetilde{\boldsymbol{\Psi}}^*\boldsymbol{\Psi}) = \boldsymbol{I}_N$. Finally, we modify the RIP-guaranteed IHT algorithm to RBOP-guaranteed ObIHT algorithm.

Ⅲ. Oblique Compressive Sensing for Anisotropic Cases

In this section, we will introduce the Oblique Compressive Sensing (OBS) concepts in terms of Restricted Biorthogonality Property (RBOP) and oblique greedy pursuit algorithms. We also show that the RBOP is a generalized form of RIP and the oblique greedy algorithms are the correspondingly modified versions of existing greedy pursuit algorithms.

3.1 Definition

Definition 1 (s-RBOC). *The s-Restricted Biorthogonal Constant (s-RBOC) $\theta_s(\boldsymbol{M})$ of $\boldsymbol{M} \in \boldsymbol{K}^{N\times N}$ is defined as the smallest δ that satisfies*

$$|\langle \boldsymbol{y}, \boldsymbol{M}\boldsymbol{x}\rangle - \langle \boldsymbol{y}, \boldsymbol{x}\rangle| \leq \delta\|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2, \tag{9}$$

for all s-sparse vectors x, y with common support.

An alternative characterization reads

$$\theta_s(\widetilde{\boldsymbol{\Psi}}^*\boldsymbol{\Psi}) = \max_{S\subseteq[N], |S|\leq s}\|E(\widetilde{\boldsymbol{\Psi}}^*\boldsymbol{\Psi}) - \boldsymbol{I}_N\|_{2\to 2}, \tag{10}$$

where

$$\|E(\widetilde{\boldsymbol{\Psi}}^*\boldsymbol{\Psi}) - \boldsymbol{I}_N\|_{2\to 2} = \max_{\boldsymbol{x}\in \boldsymbol{R}^s\setminus 0}\frac{\langle [E(\widetilde{\boldsymbol{\Psi}}^*\boldsymbol{\Psi}) - \boldsymbol{I}_N]\boldsymbol{x}, \boldsymbol{x}\rangle}{\|\boldsymbol{x}\|_2} \tag{11}$$

Definition 2 (s-RBOP). A pair of matrices $(\boldsymbol{\Psi}, \widetilde{\boldsymbol{\Psi}})$ satisfies the s-RBOP if

$$\theta_s(\widetilde{\boldsymbol{\Psi}}^*\boldsymbol{\Psi}) < c \tag{12}$$

for some constants $c \in (0,1)$. Hence, $\theta_s(\Psi^*\Psi) = \delta_s(\Psi)$, i.e., if $\tilde{\Psi} = \Psi$ the RBOP of (Ψ, Ψ) reduces to the RIP of Ψ. For the composition $\tilde{\Psi} = \tilde{A}\tilde{D}$, we find $\tilde{A}$ and $\tilde{D}$ so that $E(\tilde{A}^*A) = I_N$ and $\theta_s(\tilde{D}^*D) \ll 1$.

(i) Regarding the construction of $\tilde{A}$, we assume that $E(A^*A)$ is invertible, thus $\tilde{A} = AE(A^*A)^{-1}$. It leads to $E(\tilde{A}^*A) = I_N$.

(ii) Regarding the construction of $\tilde{D}$, we consider two cases. If D coresponds to a basis in R^N, i.e., $\mathrm{rank}(D) = \mathrm{N}$, then let $\tilde{D} = D(D^*D)^{-1}$. Otherwise, if D satisfies the RIP with certain parameter, then let $\tilde{D} = D$.

Finally, we obtain the construction of $\tilde{\Psi}$ that satisfies

$$E(\tilde{\Psi}^*\Psi) = E(\tilde{D}^*\tilde{A}^*AD) = I_N.$$

Any s columns of Ψ and $\tilde{\Psi}$ corresponding to the same indices behave like a biorthogonal basis. The small s-RBOC $\theta_s(\tilde{\Psi}^*\Psi)$ number implies that $\tilde{\Psi}^*\Psi x$ becomes close to x for all $\|x\|_0 \le s$.

3.2 Existing Reconstruction Algorithms

If Ψ satisfies the RIP, then its transpose matrix is used to compute the sparse solutions. Otherwise, in anisotropic cases, one can employ a different matrix $\tilde{\Psi}$ to get a better solution. The required property is that $\tilde{\Psi}^*\Psi x \approx x$ for any s-sparse vector $x \in R^N$. Then, we properly use designed $\tilde{\Psi}^*$ instead of Ψ^* for recovery by using modified algorithms called oblique greedy algorithms. An oblique greedy pursuit algorithm consists of the following building blocks.

(i) Find the s indices k's that maximizes $|(\tilde{\Psi}^*\Psi x)_k|$. Various greedy pursuit algorithm provide an approximation of x of guaranteed inequality $\delta_{ks}(\Psi) < c$ for some $k \in \{2,3,4\}$ and $c \in (0,1)$.

(ii) Find an approximation for the components of x on S by solving the following weighted least squares

$$\hat{z} = \arg\min_z \|\widetilde{\Psi}_S^*(\Psi x - \Psi_S z)\|_2^2, \tag{13}$$

that satisfies

$$\hat{z} = x_S + (\widetilde{\Psi}_S^*\Psi_S)^{-1}\widetilde{\Psi}_S^*\Psi_{\mathrm{supp}(x)\setminus S}\, x_{\mathrm{supp}(x)\setminus S}. \tag{14}$$

(iii) Use orthogonal matching to estimate support elements of x outside S. In this step, the oblique projection will be used instead of using the orthogonal projections in the conventional greedy algorithms.

Some modified greedy pursuit algorithms that use both Ψ and $\tilde{\Psi}$ are Oblique Thresholding (ObThres), Oblique Matching Pursuit (ObMP), and Iterative Oblique Greedy Pursuit algorithms (ObCoSaMP, ObIHT, and ObHTP). If $x \in R^N$ is an s-sparse vector and the sequence (x^n) with the measurement $y = \Psi x$ defined by the iterative greedy pursuit algorithm, then the sufficient conditions given a common form $\delta_{ks}(\Psi) < c$ for (x^n) converges to the vector x, are given in Table 1.

In the next section, we show that the RIP based guarantees of the IHT algorithm can be replaced by the similar guarantees of the corresponding ObIHT algorithms in terms of the RBOC, i.e., the condition $\theta_{3s}(\tilde{\Psi}^*\Psi) < 0.5$ is guaranteed for the sequence (x^n) with the measurement $y = \Psi x$, defined by the ObIHT algorithm, converges to the vector x, We then cover the more realistic situation of approximately sparse vectors measured with some errors, i.e., $y = \Psi x + e$.

표 1. Isotropic 경우의 선형 수렴을 위한 RBOC 조건[9]
Table 1. The RBOC conditions for linear convergence in isotropic cases[9].

Algorithm	$\delta_{ks}(\Psi) < c$
SP	$\delta_{3s} < 0.325$
CoSaOMP	$\delta_{4s} < 0.384$
IHT	$\delta_{3s} < 0.5$
HTP	$\delta_{3s} < 0.577$

Ⅳ. Analysis on Oblique Iterative Hard Thresholding Pursuit Algorithm

From previous section, we have known that the ObIHT algorithm is generalized from IHT one. It starts with an initial s-sparse vector $x^0 \in \boldsymbol{R}^N$, typically $x^0 = 0$, produces a sequence (x^n) defined by

$$x^{n+1} = H_s(x^n + \widetilde{\boldsymbol{A}}^*(y - \boldsymbol{A}x^n)). \tag{15}$$

The following properties of the pair of matrices $(\Psi, \widetilde{\Psi})$, which are generalized from the properties of RIP[13], carry over immediately to the s-RBOC matrices.

Lemma 1. *Let I_N denote the $N \times N$ identity matrix. Given vectors $u, v \in \boldsymbol{R}^N, y \in \boldsymbol{R}^m$ and an index set $S \subseteq [N] = \{1, \cdots, N\}$, we have*

$$|\langle u, (I_N - \widetilde{\Psi}^*\Psi)v\rangle| \le \theta_t \|u\|_2 \|v\|_2 \quad \text{if } \operatorname{card}(\operatorname{supp}(\mathbf{u}) \cup \operatorname{supp}(v)) \le t. \tag{16}$$

$$\|(I_N - \widetilde{\Psi}^*\Psi)v_S\|_2 \le \theta_t \|v\|_2 \quad \text{if } \operatorname{card}(S \cup \operatorname{supp}(\mathbf{v})) \le \mathrm{t}. \tag{17}$$

$$\|\widetilde{\Psi}^* y\|_2 \le \sqrt{1+\delta_s}\, \|y\|_2 \ \text{if } \operatorname{card}(S) \le s. \tag{18}$$

In practice, it is impossible to measure a signal $x \in \boldsymbol{R}^N$ with an infinite precision. Thus, the measurement vector $y \in \boldsymbol{R}^m$ is approximated by the vector $y = \Psi x$ with an error bounded by some positive constant $\epsilon > 0$. In particular, for $x \in \boldsymbol{R}^N$, we have

$$\|y - \Psi x\|_2 \le \epsilon.$$

The reconstruction procedure performs well if the reconstruction error can be controlled by the measurement error. Thus, if x is an s-sparse vector, i.e., $x = x_S$ on S, and x^n is the output of the reconstruction algorithm applied to x at the n-th iteration, we wish to have the following inequality

$$\|x^n - x_S\|_2 \le D\epsilon,$$

where the constant D depends on the sparsity. The following theorem gives us a robustness inequality to obtain robust solution when reconstructing using the ObIHT algorithm.

Theorem 1. *Suppose that $\theta_{3s}(\widetilde{\Psi}^*\Psi)$ RBOC of the pair of matrices $(\Psi, \widetilde{\Psi})$ satisfies*

$$\theta_{3s}(\widetilde{\Psi}^*\Psi) < 0.5. \tag{19}$$

If $x \in \boldsymbol{R}^N$ is an s-sparse vector, then the sequence (x^n) defined by IHT algorithm with $y = \Psi x$ converges to the vector x. Generally, if S denotes an index set of s largest (in modulus) entries of a vector $x \in \boldsymbol{R}^N$ and if $y = \Psi x + w$ for some error terms $w \in \boldsymbol{R}^m$, then

$$\|x^n - x_S\|_2 \le \rho^n \|x^0 - x_S\|_2 + \tau \|\Psi x_{\bar{S}}\|_2, n \ge 0, \tag{20}$$

where

$$\rho = 2\theta_{3s}(\widetilde{\Psi}^*\Psi) \ \text{and} \ \tau = \frac{2\sqrt{1+\delta_{2s}(\widetilde{\Psi})}}{1 - 2\delta_{3s}(\widetilde{\Psi})}. \tag{21}$$

Here, x_S is defined for any $x \in \boldsymbol{R}^n$ as

$$(x_S)_i = \begin{cases} x_i & \text{if } i \in S, \\ 0 & \text{otherwise} \end{cases}. \tag{22}$$

Proof. We desire to find a number $\rho \in [0,1)$ such that $\|x^{n+1} - x_S\|_2 \le \rho \|x^n - x_S\|_{2,} \forall n \ge 0$. Then, for some finite number of iterations, we obtain $\|x^{n+1} - x_S\|_2 \le \rho^n \|x^0 - x_S\|_{2,} \forall n \ge 0$. Taking $n \to \infty$, (21) implies $\rho^n \|x^0 - x_S\|_2 \to 0$. We assume that the s-sparse vector x^{n+1} is a better S-term approximation to

$$\begin{aligned} u^n &= x^n + \widetilde{\Psi}^*(y - \Psi x^n) \\ &= x^n + \widetilde{\Psi}^*\Psi(x_S - x^n) + \widetilde{\Psi}^*(\Psi x_{\bar{S}} + w) \end{aligned}$$

than the s-sparse vector x_S. That implies

$$\|u^n - x^{n+1}\|_2^2 \le \|u^n - x^n\|_2^2. \quad (23)$$

The left-hand side of (23) can be expanded as follows.

$$\|u^n - x^{n+1}\|_2^2 = \|(u^n - x_S) + (x_S - x^{n+1})\|_2^2. \quad (24)$$

Expanding the right-hand side of (24) and eliminating $\|u^n - x^{n+1}\|_2^2$ leads to, with $w' = \Psi x_{\bar{S}} + w$ and $U = \mathrm{supp}(\mathbf{x}) \cup \mathrm{supp}(x^n) \cup \mathrm{supp}(x^{n+1})$,

$$\begin{aligned}
&\|x^{n+1} - x_S\| \\
&\le 2\langle u^n - x_S, x^{n+1} - x_S\rangle \\
&\le 2\langle I_N - \tilde{\Psi}^* \Psi (x^n - x_S) + \tilde{\Psi}^* w', x^{n+1} - x_S\rangle \\
&\le 2\langle (I_N - \widetilde{\Psi}_U^* \Psi_U)(x^n - x_S), x^{n+1} - x_S\rangle \\
&\quad + 2\langle w', \tilde{\Psi}(x^{n+1} - x_S)\rangle \\
&\le 2\|I_N - \widetilde{\Psi}_U^* \Psi_U\|_{2\to 2}\|x^n - x_S\|_2 \|x^{n+1} - x_S\|_2 \\
&\quad + 2\|w'\|_2 \|\tilde{\Psi}(x^{n+1} - x_S)\|_2 \\
&\le 2\theta_{3s}\|x^n - x_S\|_2 \|x^{n+1} - x_S\|_2 \\
&\quad + 2\|w'\|_2 \sqrt{1 + \delta_{2s}(\tilde{\Psi})}\, \|x^{n+1} - x_S\|_2
\end{aligned}$$

Simplify by

$$\|x^{n+1} - x_S\|_2 \le 2\theta_{3s}\|x^n - x_S\|_2 + 2\sqrt{1 + \delta_{2s}(\tilde{\Psi})}\, \|w'\|_2,$$

we derive

$$\|x^{n+1} - x_S\|_2 \le 2\theta_{3s}\|x^n - x_S\|_2 + 2\sqrt{1 + \delta_{2s}(\tilde{\Psi})}\, \|w'\|_2. \quad (25)$$

If x is an s-sparse vector ($x_{\bar{S}} = 0$) and the measurements are accurate ($w = 0$), then we obtain

$$\|x^n - x_S\|_2 \le \rho^n \|x^0 - x_S\|_2. \quad (26)$$

Thus, the sequence (x^n) converges to $x = x_S$ as soon as $0 \le \rho < 1$, i.e., $\theta_{3s} < 0.5$. We can use the same technique to prove the case of exactly sparse vector measured with imperfect accuracy.

We make some following remarks on the ObIHT analysis. First, since the conventional ℓ_1-norm-based methods in anisotropic cases do not perform as well as in the isotropic cases, we present a unified way to modify s-RIP and the IHT algorithm into a new condition called s-RBOP and new algorithm called ObIHT, respectively. Second, the RBOP-guaranteed ObIHT can be applied to practical applications of CS. Comparing to the corresponding conventional algorithm, ObIIHT inherits the advantage of low computational cost and overcome the disadvantage in the anisotropic cases, thus, it often significant better. However, simulation results and analysis for the anisotropy of other measurements (except random measurements) must be considered. We will investigate these issues as future works.

Ⅴ. Conclusion

In this paper, we presented the problem of finding sparse solutions for an undetermined system of linear equations in terms of RIP. Based on this property, many computationally efficient greedy algorithms have been proposed to solve the problem of best s-sparse approximation. However, the isotropy property has only been shown under ideal assumptions, while it often violated in the setting of practical applications. Motivated by this, we study a generalized RIP called RBOP and present an extension of the IHT algorithm in terms of RBOC to solve the CS problem. Theoretical guarantees are presented and the analysis extends quite naturally.

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